Leggi di De Morgan: Sia $A_\alpha \subset \Omega$ famiglia di sottoinsiemi di Ω
$(\cup_\alpha A_\alpha)^C = \cap_\alpha A_\alpha^C, (\cap_\alpha A_\alpha)^C = \cup_\alpha A_\alpha^C$

Funzione misurabile: Dati $(\Omega,\mathcal{A})$ e $(F,\mathcal{F})$, una funzione $X:(\Omega,\mathcal{A})\to(F,\mathcal{F})$ è detta misurabile/variabile aleatoria se:
$(X\in B)\in\mathcal{A}\ \forall B\in\mathcal{F}$

Relazioni controimmagini-unioni, intersezioni o complementazioni
$X^{-1}(B^C) = (X^{-1}(B))^C$
$X^{-1}(\cup_\alpha B_\alpha) = \cup_\alpha X^{-1}(B_\alpha)$
$X^{-1}(\cap_\alpha B_\alpha) = \cap_\alpha X^{-1}(B_\alpha)$

Probabilità di un intervallo:
$P((x,y]) = F(y)-F(x)$
$P([x,y]) = F(y)-F(x^-)$
$P((x,y)) = F(y^-)-F(x)$
$P([x,y)) = F(y^-)-F(x^-)$
$P(\{x\}) = F(x)-F(x^-)$

Quantile di ordine α:
$$\begin{cases}P(X\le q_\alpha)\ge\alpha\\P(X\ge q_\alpha)\ge 1-\alpha\end{cases}\iff$$
$$\begin{cases}P(X\le q_\alpha)\ge\alpha\\P(X<q_\alpha)\le\alpha\end{cases}\implies\begin{cases}F_X(q_\alpha)\ge\alpha\\F_X(q_{\overline{\alpha}})\le\alpha\end{cases}$$

Definizione limsup, liminf:
$\liminf X_n = \lim_{n\to+\infty}(\inf_{m>n}X_m)$
$\limsup X_n = \lim_{n\to+\infty}(\sup_{m>n}X_m)$

Spazi L Si ricorda che L^1 e L^2 sono spazi vettoriali.
$X:\Omega\to R$ va reale $\in L^1 \iff \mathbb{E}[X]\in\mathbb{R}$
$x\in L^1\iff|x|\in L^1$
$X,Y\in L^p$ e $X=Y$ q.c. $\iff [X]=[Y]\in L^p$

Cauchy-Schwarz:
$X,Y\in L^2\implies \mathbb{E}[XY]|\le\sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$
$X\in L^2\implies\mathbb{E}[X]^2\le\mathbb{E}[X^2]$

Disuguaglianza di Markov:
X va reale $\implies P(|X|\ge a)\le\frac{\mathbb{E}[|X|]}{a}\quad\forall a>0$

Disuguaglianza di Chebychev:
$X\in L^2$ va reale $\implies P(|x-\mu|\ge a)\le\frac{\mathrm{Var}(x)}{a^2}$

Lemma di Fatou: Siano le va X_n e Y tali che $X_n\ge Y$ qc, $Y\in\mathcal{L}^1$. Allora $\mathbb{E}[\liminf_n X_n]\le\liminf_n\mathbb{E}[X_n]$

Tasso di fallimento: $(t>0)$
$h_X(t)=\lim_{\varepsilon\to0^+}\frac{\mathbb{P}(t<X<t+\varepsilon|X>t)}{\varepsilon}$
$h_X(t)=\frac{f_X(t)}{1-F_X(t)}$

Valore atteso e momenti

Valore atteso: $\mathbb{E}[h(X)]=\int_\Omega h(X(\omega))I(d\omega)=\int_\Omega h(X(\omega))P(d\omega)=\int_\mathbb{R}h(x)P^X(dx)=\int_\mathbb{R}h(x)f_X(x)dx$
Nel caso discreto: $\mathbb{E}[h(x)]=\sum_k h(x_k)p_k$

Varianza e covarianza:
$\mathrm{Var}(X)=\sigma_X^2=\mathbb{E}[X^2]-\mathbb{E}[X]^2$
$\mathrm{Cov}(X,Y)=\sigma_{XY}=\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]$
$\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)+2\,\mathrm{Cov}(X,Y)$
$\mathrm{Var}(aX+b)=a^2\,\mathrm{Var}\,X$
$\mathrm{Cov}(a,X)=0$
$\mathrm{Cov}(aX+bY,Z)=a\,\mathrm{Cov}(X,Z)+b\,\mathrm{Cov}(Y,Z)$
$\mathrm{Cov}(X,Y)=\rho\sigma_X\sigma_Y$

Coefficiente di correlazione lineare:
$\rho=\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$

Trasformazioni affini: Dato $Y=AX+b$,
$\mathbb{E}[Y]=A\mathbb{E}[X]+b$,
$\mathrm{Var}[Y]=A\,\mathrm{Var}[X]A^T=C_Y=AC_XA^T$

Vettori aleatori

Sia X vettore aleatorio:
$\mathbb{E}[h(X)]=\int_\Omega h(X)d\mathbb{P}=\int_{\mathbb{R}^n}h(x)dP^X$
$$=\begin{cases}\sum_{x\in S}h(x)p(x) & X\text{ è discreto}\\ \int_{\mathbb{R}^n}h(x)f(x)dx & X \text{ è continuo}\end{cases}$$
Per calcolare la funzione di ripartizione di una variabile specifica:
Continuo: $f_k(x_k)=\int f(x_1,\dots,n)dx_1\dots dx_{k-1}dx_{k+1}\dots dx_n$ (integrale su tutte le componenti che non ci interessano, ovvero tutte tranne x_k)
Discreto: gli integrali diventano sommatorie.
X è un vea continuo se C è invertibile.

Trasformazione vea continuo: Caso $\mathbb{R}$:
$Y=g(X)\iff y=g(x)\rightsquigarrow x=h(y)$
$\implies f_Y(y)=f_X(x=h(y))\cdot|h'(y)|$

Caso $\mathbb{R}^2$:
$$\begin{pmatrix}U\\V\end{pmatrix}=g\begin{pmatrix}X\\Y\end{pmatrix}=\begin{cases}g_1(x,y)\\g_2(x,y)\end{cases}$$
$$\begin{pmatrix}X\\Y\end{pmatrix}=h\begin{pmatrix}U\\V\end{pmatrix}=\begin{cases}h_1(u,v)\\h_2(u,v)\end{cases}$$
$$J_h=\begin{pmatrix}\partial_u h_1 & \partial_v h1\\ \partial_u h_2 & \partial_v h_2\end{pmatrix}=\begin{pmatrix}\nabla h_1\\ \nabla h_2\end{pmatrix}$$
$$\implies f_{U,V}(u,v)=f_{X,Y}\left(\begin{pmatrix}x\\y\end{pmatrix}=h\begin{pmatrix}u\\v\end{pmatrix}\right)\cdot|J_h|$$

Vettori aleatori gaussiani

Dato $X=(X_1,\dots,X_n)\sim\mathcal{N}(\mu,C)$ dove $\mu\in\mathbb{R}^n, C\in\mathbb{R}^{n\times n}, C>0$, vale che X è un vettore gaussiano ($\langle\cdot|\cdot\rangle$ è il prod. scalare) $\iff$ $\varphi_{(X_1,\dots,n)}(u)=\exp\{i\langle u|\mu\rangle-\frac12\langle u|Cu\rangle\}\iff$ $\langle a|X\rangle\sim\mathcal{N}\ \ \forall a\in\mathbb{R}^n$
Quindi $f_X(\mathbf{x})=\frac{1}{\sqrt{(2\pi)^n\det(C)}}\exp\{-\frac12\langle\mathbf{x}-\mu|C^{-1}(\mathbf{x}-\mu)\rangle\}$

Proprietà: 1. $X_k\sim\mathcal{N}(\mu_k,C_{kk})$
2. $S_X=\mathrm{Im}(C)+\mu=\mathrm{Col}(C)+\mu=[\mathrm{Ker}(C)]^\perp+\mu$
3. X Continuo $\iff\det(C)\neq0\iff C>0$
4. $X_i\perp\!\!\!\perp X_j\iff\mathrm{Cov}(X_i,X_j)=0$
5. $X_i\sim\mathcal{N}\not\Longrightarrow X\sim\mathcal{N}$, solo se sono anche $\perp\!\!\!\perp$

Trasformazioni affini:
$Y=AX+b,\quad Y\sim\mathcal{N}(A\mu+b,ACA^T)$
$X\sim\mathcal{N}(\mu_X,\sigma_X^2),\quad Y\sim\mathcal{N}(\mu_Y,\sigma_Y^2)$
$aX+bY\sim\mathcal{N}(a\mu_X+b\mu_Y,a^2\sigma_X^2+b^2\sigma_Y^2+2ab\sigma_{XY})$
Da verificare: $a^2\sigma_X^2+b^2\sigma_Y^2+2ab\sigma_{XY}\ge0$

Caso bidimensionale:
$$\begin{bmatrix}X\\Y\end{bmatrix}\sim\mathcal{N}\left(\begin{bmatrix}\mu_X\\\mu_Y\end{bmatrix},\begin{bmatrix}\sigma_X^2&\mathrm{Cov}(X,Y)\\\mathrm{Cov}(X,Y)&\sigma_Y^2\end{bmatrix}\right)$$
$$\varphi_{(X,Y)}(u,v)=\exp\{i(\mu_Xu+\mu_Yv)-\frac12(u^2\sigma_X^2+2\sigma_{XY}uv+v^2\sigma_Y^2)\}$$
In questo caso la proprietà (3) può anche essere espressa come:
$$\begin{bmatrix}X\\Y\end{bmatrix}\text{ Cont. }\iff\det C\neq0\iff\begin{cases}\sigma_X>0\\\sigma_Y>0\\|\rho_{X,Y}|<1\end{cases}$$
$$f_{(X,Y)}(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}-2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\}$$

Funzioni caratteristiche

Funzione complessa in corrispondenza biunivoca con la legge, la quale è da essa caratterizzata.
$$\varphi_X(u)=\int_{\mathbb{R}^n}e^{i\langle u|X\rangle}P^X(dx)=\mathbb{E}\left[e^{i\langle u|X\rangle}\right]=\int_\Omega e^{i\langle u|X\rangle}d\mathbb{P}=\Big|_{(\text{su }\mathbb{R})}\int_\mathbb{R}e^{iux}\cdot f_X(x)dx$$

Funzione caratteristica e momenti:
$\frac{\partial^m}{\partial u_{k1}\cdots\partial u_{km}}\varphi(0)=i^m\mathbb{E}[X_{k1}\cdots X_{km}]$
$\mathbb{E}[X_k]=\frac1i\frac{\partial\varphi(0)}{\partial u_k}$
$\mathbb{E}[X_k^2]=-\frac{\partial^2\varphi(0)}{\partial u_k^2},\ \mathbb{E}[X_kX_j]=-\frac{\partial^2\varphi(0)}{\partial u_k\partial u_j}$
$\mathbb{E}(X^k)=(-i)^k\frac{d^k}{dt^k}\varphi_X(t)\Big|_{t=0}$

Funzione generatrice dei momenti:
$m_X(t)=\mathbb{E}(e^{tX})\implies\mathbb{E}(X^k)=\frac{d^k}{dt^k}m_X(t)\Big|_{t=0}$

Trasformazioni affini e indip:
$Y=AX+b\implies\varphi_Y(u)=e^{i\langle u|b\rangle}\varphi_X(A^Tu)$
$X\perp\!\!\!\perp Y\implies\varphi_{X+Y}(u)=\varphi_X(u)\cdot\varphi_Y(u)$
$X_i\perp\!\!\!\perp\implies\varphi_{X_1,\dots,X_n}(u_1,\dots,u_n)=\prod\varphi_{X_i}(u_i)$

Prob. e leggi condizionate

Probabilità condizionata:
$P(A|B)=\frac{P(A,B)}{P(B)}$
$P(A)=\sum_nP(A\cap E_n)=\sum_nP(A|E_n)P(E_n)$

Formula di Bayes:
$P(E_k|A)=\frac{P(A|E_k)P(E_k)}{\sum_nP(A|E_n)P(E_n)}$

Leggi condizionate:
$Y|X=x\sim f_{Y|X}(\cdot|x)$
$f_{(X,Y)}(x,y)=f_{(Y|X)}(y,x)\cdot f_X(x)$
$\mathbb{E}[Y|X=x]=m(x)=\int_{S_Y}yf_{Y|X}(y,x)dy$
$\mathrm{Var}(Y|X=x)=q^2(x)=\int_{S_Y}(y-m(x))^2f_{Y|X}(y,x)dy$
Nel caso di va discrete è sufficiente sostituire P ad f e svolgere gli integrali come sommatorie.

Valore atteso condizionato: Se $Y\in L^1$ allora $\mathbb{E}[Y]=\mathbb{E}[\mathbb{E}[Y|X]]$

Varianza condizionata:
$\mathrm{Var}(Y|X)=\mathbb{E}[Y^2|X]-\mathbb{E}[Y|X]^2$
$\mathrm{Var}(Y)=\mathrm{Var}(\mathbb{E}[Y|X])+\mathbb{E}[\mathrm{Var}[Y|X]]$

Vettori Gaussiani condizionati 2-dim:
$X\sim\mathcal{N}(\mu_X,\sigma_X),\ Y\sim\mathcal{N}(\mu_Y,\sigma_Y)$
$$\begin{bmatrix}X\\Y\end{bmatrix}\sim\mathcal{N}\left(\begin{bmatrix}\mu_X\\\mu_Y\end{bmatrix},\begin{bmatrix}\sigma_X^2&\rho\sigma_X\sigma_Y\\\rho\sigma_X\sigma_Y&\sigma_Y^2\end{bmatrix}\right)$$
Se $\sigma_X=0\implies X\sim\mathcal{N}(\mu_X,0)=\delta_{\mu_X}$
$\implies\mathbb{P}(X=\mu_X)=1$
$\implies Y|X=\mu_X\sim Y\sim\mathcal{N}(\mu_Y,\sigma_Y^2)$
Se $\sigma_X>0\implies Y|X=s\sim\mathcal{N}(m(s),q^2)$
$m(s)=\mu_Y+\frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(s-\mu_X)$
$=\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(s-\mu_X)$
$q^2=\mathrm{Var}(Y)-\frac{\mathrm{Cov}(X,Y)^2}{\mathrm{Var}(X)}=\sigma_Y^2(1-\rho^2)$

Vettori Gaussiani condizionati n-dim:
$X\sim\mathcal{N}(\mu_X,C_X)$ n-dim
$Y\sim\mathcal{N}(\mu_Y,C_Y)$ m-dim
$$\begin{bmatrix}X\\Y\end{bmatrix}\sim\mathcal{N}\left(\begin{bmatrix}\mu_X\\\mu_Y\end{bmatrix},\begin{bmatrix}C_X&C_{XY}\\C_{YX}&C_Y\end{bmatrix}\right)$$
$C_{XY}=\mathrm{Cov}(X_i,Y_j)\in\mathbb{R}^{n\times m}$
$C_{YX}=C_{XY}^T=\mathrm{Cov}(X,Y)\in\mathbb{R}^{m\times n}$
$\det(C_X)>0\implies Y|X=s\sim\mathcal{N}(m(s),Q)$
$m(k)=\mathbb{E}[Y|X=k]=\mu_Y+C_{YX}C_X^{-1}(k-\mu_X)$
$Q=\mathrm{Var}(Y|X=k)=C_Y-C_{YX}C_X^{-1}C_{XY}$
$W=(Y|X=k)\sim\mathcal{N}(m(k),Q)$

Distribuzioni

Delta di Dirac (δ_n)
$P(X=n)=1\quad\varphi(u)=e^{inu}$

Continua uniforme ($U(a,b)$)
$f(x)=\frac{1}{b-a}\mathbb{1}_{(a,b)}(x)\quad F(x)=\frac{x-a}{b-a}$
$\mathbb{E}[X]=\frac{a+b}{2},\quad\mathrm{Var}[X]=\frac{(b-a)^2}{12}$
$\varphi(u)=e^{i\frac{a+b}{2}u}\sin\left(\frac{b-a}{2}u\right)/\left(\frac{b-a}{2}u\right)$
con $-\infty<a<b<+\infty$.

Bernoulli ($\mathrm{Be}(p)$) Misura l'esito di un esperimento vero-falso. Supporto: $\{0,1\}$
$\mathrm{Be}(p)\iff\varphi_X(u)=pe^{iu}+1-p$, con $p\in[0,1]$
$\mathbb{E}[X]=p,\ \mathrm{Var}[X]=p(1-p)$

Binomiale ($\mathrm{Bi}(n,p)$) Somma di n $\mathrm{Be}(p)$. Supporto: $\{0,1,2,\dots\}$
$p_X(k)=\binom nk p^k(1-p)^{n-k},\binom nk=\frac{n!}{k!(n-k)!}$
$\mathbb{E}[X]=np,\ \mathrm{Var}[X]=np(1-p)$
$\mathrm{Bi}(n,p)\iff\varphi(u)=(pe^{iu}+1-p)^n$,
con $p\in[0,1]$ e $n\in\mathbb{N}$.

Geometrica ($\mathcal{G}(p)$) Numero di fallimenti prima di un successo in un processo di Bernoulli. Priva di memoria. Supporto: $\{1,2,3\dots\}$
$p_X(k)=p(1-p)^{k-1}$
$F(k)=P(X\le k)=1-P(X\ge k+1)$
$=1-(1-p)^k$
$\mathbb{E}[X]=\frac1p,\ \mathrm{Var}[X]=\frac{1-p}{p^2}$
$\mathcal{G}(p)\iff\varphi(u)=\frac{pe^{iu}}{1-e^{iu}(1-p)}$

Geometrica traslata: $P(W=k)=p(1-p)^k$

Ipergeometrica ($\mathcal{H}(n,h,r)$) Descrive l'estrazione senza reimmissione di palline da un'urna con n palline di cui h del tipo X e $n-h$ del tipo Y. La probabilità di ottenere k palline del tipo X estraendone r dall'urna è
$p_X(k)=\frac{\binom hk\binom{n-h}{r-k}}{\binom nr}$
per $\max\{0,h+r-n\}\le k\le\min\{r,h\}$
$\mathbb{E}[X]=\frac{rh}{n},\ \mathrm{Var}[X]=\frac{h(n-h)r(n-r)}{n^2(n-1)}$

Poisson ($\mathcal{P}(\lambda)$) Legge degli eventi rari. Limite delle distribuzioni binomiali con $\lambda=np$. Supporto: $\{0,1,2,\dots\}$
$p_X(k)=e^{-\lambda}\frac{\lambda^k}{k!},\ \mathbb{E}[X]=\lambda,\ \mathrm{Var}[X]=\lambda$
$\mathcal{P}(\lambda),\lambda>0\iff\varphi(u)=\exp\{\lambda(e^{iu}-1)\}$

Normale ($\mathcal{N}(\mu,\sigma^2)$)
$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$
$\mathbb{E}[X]=\mu,\ \mathrm{Var}[X]=\sigma^2,\ \mathbb{E}[(X-\mu)^4]=3\sigma^4$
$F(x)=P(X\le x)=P\left(\frac{X-\mu}{\sigma}\le\frac{x-\mu}{\sigma}\right)$
$=P\left(Z\le\frac{x-\mu}{\sigma}\right)=\Phi\left(\frac{x-\mu}{\sigma}\right)$
con $Z=\frac{X-\mu}{\sigma}\sim\mathcal{N}(0,1)$
$\mathcal{N}(\mu,\sigma^2)\iff\varphi(u)=\exp\left\{iu\mu-\frac{\sigma^2}{2}u^2\right\}$
$m_X(t)=\exp\left\{\mu t+\frac{\sigma^2}{2}t^2\right\}$

$\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^p\right]=\begin{cases}0&p\text{ dispari}\\(p-1)!!&p\text{ pari}\end{cases}$
ma σ^p si può anche scalare fuori. Per $k>0$ vale
$X\sim\mathcal{N}(0,\sigma^2)\implies\mathbb{E}[e^{-kX^2}]=(1+2k\sigma^2)^{-1/2}$
$\mathbb{E}(|X|)=\sqrt{\frac2\pi}\sigma\exp\{-\frac{\mu^2}{2\sigma^2}\}+\mu[1-2\Phi(-\frac\mu\sigma)]$
$X=\lim(X_n\sim\mathcal{N}(\mu_n,\sigma_n^2))\sim\mathcal{N}(\lim\mu_n,\lim\sigma_n^2)$

Lognormale ($\log\mathcal{N}(\mu,\sigma^2)$), $X=e^\mathcal{N}$
$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}\mathbb{1}_{(0,+\infty)}$
$F(x)=\Phi_{(\mu,\sigma)}(\ln x)$
$\mathbb{E}[X]=e^{\mu+\sigma^2/2},\ \mathrm{Var}[X]=e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$

Chi-quadro ($\chi^2(k)$) Somma di k normali standard $\mathcal{N}(0,1)$ al quadrato.
$f(x)=\frac{1}{2^{k/2}\Gamma(k/2)}x^{k/2-1}e^{-x/2}\mathbb{1}_{(0,+\infty)}$
$\mathbb{E}[X]=k,\ \mathrm{Var}[X]=2k$
$\chi^2(n)=\Gamma\left(\frac n2,\frac12\right)$
$k\cdot\chi^2(n)\sim\Gamma\left(\frac n2,\frac1{2k}\right)$
$\chi^2(k)\iff\varphi(u)=(1-2iu)^{-k/2}$

T di Student ($T(n)$)
$T=Z/\sqrt{\frac Qn}, Z\sim\mathcal{N}(0,1),Q\sim\chi^2(n),Z\perp\!\!\!\perp Q$
$f(x)=\frac{\Gamma(\frac{n+1}2)}{\Gamma(\frac n2)\sqrt{\pi n}}\cdot\frac{1}{(1+\frac{x^2}{n})^{\frac{n+1}2}}$
$\mathbb{E}[T]=0$ se $n>1$ oppure indefinito.
$\mathrm{Var}(T)=\frac{n}{n-2}$ se $n>2$ oppure indefinita.

Esponenziale ($\mathcal{E}(\lambda)$) Durata di vita di un fenomeno. Priva di memoria. $\lambda>0$
$f(x)=\lambda e^{-\lambda x}\mathbb{1}_{(0,+\infty)}(x)\quad F(x)=1-e^{-\lambda x}$
$\mathbb{E}[X^n]=\frac{n!}{\lambda^n},\ \mathrm{Var}[X]=\frac1{\lambda^2}$
$\mathcal{E}(\lambda)\iff\varphi(u)=\frac{\lambda}{\lambda-iu}$

Gamma ($\Gamma(\alpha,\lambda)$) Somma di va indipendenti con dist. esponenziale. $\lambda>0,\alpha>0$
$f(x)=\frac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}\mathbb{1}_{(0,+\infty)}(x)$
$F(x)=\left(1-\sum_{k=0}^{\alpha-1}e^{-\lambda x}\frac{(\lambda x)^k}{k!}\right)=\frac{\gamma(\alpha,\lambda x)}{\Gamma(\alpha)}$
con $x\in[0,+\infty)$ e α intero.
$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$
$\Gamma(\alpha+1)=\alpha\Gamma(\alpha)\quad\Gamma(n+1)=n!$
$\Gamma(1)=1\quad\Gamma\left(\frac12\right)=\sqrt\pi\quad\mathbb{E}[X]=\frac\alpha\lambda\quad\mathrm{Var}[X]=\frac\alpha{\lambda^2}$
$Z^2\sim\Gamma\left(\frac12,\frac12\right)=\chi^2(1)$
$\mathbb{E}[X^k]=\frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{\lambda^k}$
con $k\in\mathbb{Z},\alpha+k>0$
$c>0,X\sim\Gamma(\alpha,\lambda)\implies Y=cX\sim\Gamma\left(\alpha,\frac\lambda c\right)$
$\Gamma(\alpha,\lambda)\iff\varphi(u)=\left(\frac{\lambda}{\lambda-iu}\right)^\alpha$

Weibull ($W(\lambda,k)$)
$\lambda>0,k>0$
$f(x)=\frac{k}{\lambda^k}x^{k-1}e^{-(x/\lambda)^k}\mathbb{1}_{(0,+\infty)}(x)$
$F(x)=e^{-(\frac x\lambda)^k}\quad\mathbb{E}[X]=\frac\lambda k\Gamma\left(\frac1k\right)$
$\mathrm{Var}[X]=\frac{\lambda^2}{k^2}\left[2k\Gamma\left(\frac2k\right)-\Gamma^2\left(\frac1k\right)\right]$
Imponendo $k=1$ è un'esponenziale.

Cauchy ($\mathcal{C}(x_0,\gamma)$) $\gamma>0,x_0\in\mathbb{R}$
$f(x)=\frac{1}{\pi\gamma}\left[\frac{\gamma^2}{(x-x_0)^2+\gamma^2}\right]$
$F(x)=\frac1\pi\arctan\left(\frac{x-x_0}{\gamma}\right)+\frac12$
$\mathcal{C}(x_0,\gamma)\iff\varphi(u)=e^{ix_0u-\gamma|u|}$
La distribuzione di Cauchy non ha né valore atteso né varianza, $\sim t(1)$

Convergenza di VA

Convergenza monotona: Siano le va X_n e X tali che $0\le X_n\le+\infty, X_n\uparrow X$ qc, allora:
$\mathbb{E}[X_n]\xrightarrow{n}\mathbb{E}[X]$, cioè $\mathbb{E}[\lim_nX_n]=\lim_n\mathbb{E}[X_n]$

Convergenza dominata: Siano le va reale X_n,X,Y tali che $X_n\xrightarrow{\text{qc}}X,|X_n|\le Y$ qc $\forall n,Y\in\mathcal{L}^1$, allora:
$X_n\in\mathcal{L}^1,X\in\mathcal{L}^1,\mathbb{E}[X_n]\xrightarrow{n}\mathbb{E}[X]$

Certa: Simile ad una convergenza puntuale, $X_n\to X\ \forall\omega\in\Omega$

Quasi certa: $\mathbb{P}(X_n\to X)=1$ ovvero $A=(X_n\to X)\in\mathcal{A}$ e $\mathbb{P}(A)=1$. Questo significa che la conv. vale a meno di insiemi di misura nulla.

Negli spazi L^p: $X_n\xrightarrow{L^p}X$ se $X_n\in L^p\ \forall n,X\in L^p$ e $\mathbb{E}[|X_n-X|^p]\to0$
ie if $\|X_n-X\|\to0$ dove $\|U\|_{L^p}=[\mathbb{E}(|U|^p)]^{1/p}$
Dato $p\ge q\ge1$, conv. in $L^p\implies$ conv. in L^q.
Convergono i momenti
$\mathbb{E}[|X_n|^p]\xrightarrow{L^p}\mathbb{E}[|X|^p]$

Quindi per $p=1$ e $p=2$ vale che:
$\mathbb{E}[X_n]\to\mathbb{E}[X],\quad\mathrm{Var}(X_n)\to\mathrm{Var}(X)$

Probabilità: $X_n\xrightarrow{\mathbb{P}}X$ se $\forall\varepsilon>0\ \mathbb{P}(|X_n-X|\ge\varepsilon)\to0$
Esiste una sottosuccessione di X_n che converge qc a X. Se X appartiene a L^p ed è possibile trovare una Y tale che $|X_n|\le Y$ allora questa converge anche in L^p.

Debole: Siano $\mathbb{P}_n$ e $\mathbb{P}$ prob su $(\mathbb{R},\mathcal{B})$. Allora $\mathbb{P}_n\xrightarrow{\text{deb}}\mathbb{P}$ se $\int_\mathbb{R}h\,d\mathbb{P}_n\to\int_\mathbb{R}h\,d\mathbb{P}\ \forall h\in C_b(\mathbb{R})$

In legge o distribuzione: $X_n\xrightarrow{\mathcal{L}}X$ se $P^{X_n}\xrightarrow{\text{deb}}P^X$
Slutsky: $X_n\xrightarrow{\mathcal{L}}c\implies X_n\xrightarrow{\mathbb{P}}c$

Criteri conv. in legge: *Discreto:* $S=S_X\cup S_{X_n}$ [con $S\subseteq\mathbb{N}$, non denso],
$\mathbb{P}(X_n=s)\to\mathbb{P}(X=s)\ \forall s\in S\implies X_n\xrightarrow{\mathcal{L}}X$.
Se $S\subseteq\mathbb{Z}$ o S finito, vale anche opposto.
Continuo: $f_n\xrightarrow{\text{qo}}f\implies X_n\xrightarrow{\mathcal{L}}X$
$\iff F_n(t)\to F(t)\ \forall t\in S$ dove F è cont.
Levy (corollario): $X_n\xrightarrow{\mathcal{L}}X\iff\varphi_{X_n}\to\varphi_X$
X_n conv. in $\mathcal{L}\iff\varphi_{X_n}(u)\to\varphi_X(u)$ e φ_X è cont. in 0.

Relazione tra le convergenze:
$$\begin{tikzcd}
 & & \text{q.c.} \arrow[d, shift right] \\
L^p \arrow[r, "q<p"] & L^q \arrow[r, shift right] & \mathbb{P} \arrow[l, shift right, "|X_n|\le Y,\ Y\in L^q"'] \arrow[u, shift right, dashed, "\exists\text{ sottosucc}"'] \arrow[d, shift left] \\
 & & \mathcal{L} \arrow[u, shift left, "X=c\text{ q.c.}"]
\end{tikzcd}$$

Proprietà: Se $X_n\to X$ e $Y_n\to Y$, allora a seconda della convergenza possono valere alcune o tutte le seguenti proprietà:
1. $aX_n\to aX$ (qc, $L^p,\mathbb{P},\mathcal{L}$)
2. $h(X_n)\to h(X)$ (qc, $\mathbb{P},\mathcal{L}$)
3. $X_n+Y_n\to X+Y$ (qc, $L^p,\mathbb{P}$)
4. $X_nY_n\to XY$ (qc, $\mathbb{P}$)
5. $Y,Y_n\neq0\ \forall\omega\implies\frac{X_n}{Y_n}\to\frac XY$ (qc, $\mathbb{P}$)

Se Y converge a una costante in legge valgono anche la 3, 4 e 5 (teorema di Slutsky)

LGN: Dati $X_n\in L^1$ iid e $\mu\in\mathbb{R}$, allora $\mathbb{E}[X_n]=\mu\iff\overline{X}_n\xrightarrow{\text{qc}}\mu$ e $\overline{X}_n\xrightarrow{L^1}\mu$

TCL: Dati $X_n\in L^2$ iid, $\mu=\mathbb{E}[X_n]\quad\sigma^2=\mathrm{Var}(X_n)>0$, allora:
$(\overline{X}_n-\mu)/\left(\frac{\sigma}{\sqrt n}\right)\xrightarrow{\mathcal{L}}N(0,1)$

Velocità di convergenza: X_n converge verso a con vel. v_n se: $v_n(X_n-a)\xrightarrow{\mathcal{L}}T\nsim\delta_0$

Asintotica normalità: Se $T\sim\mathcal{N}(0,q)$, con $q>0$ allora: $X_n\sim AN(a,q/v_n^2)$

Metodo Delta 1: Sia $\overline{X}_n\xrightarrow{\mathcal{L}}\mathcal{N}(\mu,\frac{\sigma^2}{n})$
Se $g:\mathbb{R}\to\mathbb{R},g\in C^1(\mathbb{R}),g'(\mu)\neq0$
$\implies g(\overline{X}_n)\xrightarrow{\mathcal{L}}\mathcal{N}(g(\mu),\frac{\sigma^2}{n}[g'(\mu)]^2)$

Risultati notevoli

Media campionaria: $\overline{X}_n=\frac1n\sum_{k=1}^nX_k$
1. $\mathbb{E}[\overline{X}_n]=\mu\ \forall n$ (correttezza)
2. $\mathrm{Var}(\overline{X}_n)=\frac{\sigma^2}{n}\xrightarrow{n}0$ (ovvero $\overline{X}_n\xrightarrow{L^2}\mu$)
3. $\overline{X}_n\xrightarrow{\text{qc}}\mu$ per la LGN (consistenza)
4. $\frac{\overline{X}_n-\mu}{\sigma/\sqrt n}\xrightarrow{\mathcal{L}}\mathcal{N}(0,1)$ per il TCL.

Varianza campionaria:
$S_n^2=\frac1{n-1}\sum_{k=1}^n(X_k-\overline{X}_n)^2$
1. $\mathbb{E}[S_n^2]=\sigma^2\ \forall n$ (corretto)
3. $S_n^2\xrightarrow{\text{qc}}\sigma^2$ (consistente)
4. $(S_n^2-\sigma^2)/\left(\sqrt{\frac{\mu_4-\sigma^4}{n}}\right)\xrightarrow{\mathcal{L}}N(0,1)$

ovvero $S_n^2\sim AN\left(\sigma^2,\frac{\mu_4-\sigma^4}{n}\right)$
dove $\mu_4=\mathbb{E}[(X_n-\mu)^4]\neq\sigma^4$

Prodotto di convoluzione:
$Z=X+Y,\ P^Z=P^X*P^Y$
$f_Z(z)=(f_X*f_Y)(z)=\int_\mathbb{R}f_X(z-t)f_Y(t)dt$
Caso discreto:
$\mathbb{P}(U=X+Y=k)=p_U(k)=\sum_{h=0}^k\mathbb{P}(X=h,Y=k-h)$

Somma di distribuzioni: *Bernoulliane:* $X_1,\dots,X_n\sim\mathrm{Be}(p)$ indip.
$\implies X_1+\dots+X_n\sim\mathrm{Bi}(n,p)$
Binomiali: $X_1\sim\mathrm{Bi}(n_1,p),X_2\sim\mathrm{Bi}(n_2,p)$ indip.
$\implies X_1+X_2\sim\mathrm{Bi}(n_1+n_2,p)$
Gaussiane: $X_1,\dots,X_n\sim\mathcal{N}(0,1)$ indip.
$\implies X_1+\dots+X_n\sim\mathcal{N}(0,n)$
$\implies X_1^2+\dots+X_n^2\sim\chi^2(n)$
Esponenziali: $X_1,\dots,X_n\sim\mathcal{E}(\lambda)$ indip.
$\implies X_1+\dots+X_n\sim\Gamma(n,\lambda)$
Gamma: $X\sim\Gamma(\alpha,\lambda),\ Y\sim\Gamma(\beta,\lambda)$ indip.
$\implies X+Y\sim\Gamma(\alpha+\beta,\lambda)$
Poisson: $X\sim\mathcal{P}(\lambda_X),\ Y\sim\mathcal{P}(\lambda_Y)$ indip.
$\implies X+Y\sim\mathcal{P}(\lambda_X+\lambda_Y)$

Serie: (per sicurezza)
$\sum_{k=0}^\infty q^k=\frac1{1-q},\ |q|<1$
$\sum_{k=0}^\infty kq^k=\frac{q}{(1-q)^2},\ |q|<1$
$\sum_{k=0}^N q^k=\frac{1-q^{N+1}}{1-q},\ |q|<1$
$e^x=\lim_{n\to+\infty}\left(1+\frac xn\right)^n=\sum_{k=0}^\infty\frac{x^k}{k!}$
$\sum_{i=1}^ni=\frac{n(n+1)}2$
$\sum_{i=1}^ni^2=\frac{n(n+1)(2n+1)}6$
$\sum_{i=1}^\infty\frac{x^i}{i}=\ln\left(\frac1{1-x}\right),\ |x|\le1,\ x\neq1$
$\sum_{i=0}^n\binom ni a^{(n-i)}b^i=(a+b)^n$
$\sum_{i=1}^\infty ix^i=\frac{x}{(1-x)^2}$

Recap from SDE

$\mathcal{F}_1,\dots,\mathcal{F}_n\subseteq\mathcal{F}$ are $\perp\!\!\!\perp$ σ-algebras $\iff$ $\forall A_i\in\mathcal{F}_i\quad\mathbb{P}(\bigcap_{i=1}^nA_i)=\prod_{i=1}^n\mathbb{P}(A_i)$
$X_1,\dots,X_n$ of $(E_i,\mathcal{E}_i)$ valued rvs are $\perp\!\!\!\perp\iff$ $\forall B_i\in\mathcal{E}_i\quad\mathbb{P}(\cap_i(X_i\in B_i))=\prod_i\mathbb{P}((X_i\in B_i))$
Now passing to gaussians:
$\varphi_X(\xi)=\varphi_Y(\xi)\ \ \forall\xi\in\mathbb{R}(\text{or }\mathbb{R}^n)\iff X\sim Y$
X va $\mathcal{N}\iff\phi_X(u)=e^{\left[iu\mu-\sigma^2\frac{u^2}{2}\right]}$
X vea $\mathcal{N}\iff\phi_X(\mathbf{u})=e^{\left[i\langle\mathbf{u},\mu\rangle-\frac12\langle\Sigma\mathbf{u},\mathbf{u}\rangle\right]}$
(1) $\mathbf{X}\sim\mathcal{N}(\mu,\Sigma)$ and $X\sim\mathcal{N}(\mu,\sigma^2)\implies$
$g(\mathbf{X})=\mathbf{A}\mathbf{X}+b\sim\mathcal{N}(A\mu+b,A\Sigma A^T)$
$g(X)=ax+b\sim\mathcal{N}(a\mu+b,a^2\sigma^2)$
(2) $\mathbf{X}$ is a gaussian n-dim rvec $\implies$ every linear comb or transf of its component is a $\mathcal{N}$ rv or rvec
(3) $X\perp\!\!\!\perp Y$ rv or rvec $\implies$ (X,Y) is a gaussian rvec
(4) $\perp\!\!\!\perp$ through Cov holds only in $\mathcal{N}$ rvec
Morale: tutte le belle proprietà si hanno con vettori gaussiani.

Strumentopoli misteriosi

X va positiva e continua:
$\mathbb{E}[X]=\int_0^{+\infty}(1-F(x))dx=\int_0^{+\infty}\mathbb{P}(X>k)dx$
nel caso discreto vale $\mathbb{E}[X]=\sum_{k=0}^{+\infty}\mathbb{P}(X>k)$
va indip. $\iff$ fattorizzano: densità, φ, $\mathbb{E}$
Se (X_1,X_2) indip. $\implies\mathrm{Cov}(X_1,X_2)=0$
Ricondursi ai punti precedenti, usare trasformazioni

Funzione di ripartizione
$F_{\max}(t)=\prod F(t)$
$F_{\min}(t)=1-\prod(1-F(t))$

Campioni gaussiani $X_1\dots X_n$ iid $X_k\sim\mathcal{N}(\mu,\sigma^2),\ \mu\in\mathbb{R},\ \sigma^2>0$:
1. $\overline{X}_n\sim\mathcal{N}(\mu,\sigma^2/n)$
2. $S^2\sim\frac{\sigma^2}{n-1}\chi^2(n-1)$
3. $\overline{X}_n\perp\!\!\!\perp S^2$
4. $\frac{\overline{X}_n-\mu}{\sqrt{S^2/n}}\sim t(n-1)$

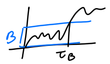
Convergenza integrali
$$\int_a^b\frac{1}{(x-a)^p}\implies\begin{cases}\text{converge se }p<1\\\text{diverge se }p\ge1\end{cases}$$
$$\int_{\alpha\in(0,1)}^{+\infty}\frac1{x^p}\implies\begin{cases}\text{converge se }p>1\\\text{diverge se }p\le1\end{cases}$$
$$\int_0^{\alpha>1}\frac{1}{x^a|\ln x|^b}\implies\begin{cases}\text{converge}\begin{cases}a<1&\forall b\in\mathbb{R}\\a=1&b>1\end{cases}\\\text{diverge}\begin{cases}a>1&\forall b\in\mathbb{R}\\a=1&b\le1\end{cases}\end{cases}$$
$$\int_{\alpha>1}^{+\infty}\frac{1}{x^a(\ln x)^b}\implies\begin{cases}\text{conerge}\begin{cases}a>1&\forall b\in\mathbb{R}\\a=1&b>1\end{cases}\\\text{diverge}\begin{cases}a<1&\forall b\in\mathbb{R}\\a=1&b\le1\end{cases}\end{cases}$$
$$\int_1^{\alpha>1}\frac{1}{(\ln x)^p}\implies\begin{cases}\text{converge se }p<1\\\text{diverge se }p\ge1\end{cases}$$
$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}=\frac1{\det A}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$$

STOCH PROCESSES

$\mathcal{G}_t=\sigma(X_s: 0\le s\le t)$ gives adaptiveness for free
$\mathcal{F}_\infty=\bigvee_{t\in T}\mathcal{F}_t$, $\overline{\mathcal{G}}_t=\mathcal{G}_t\vee\mathcal{N}_{\mathcal{F}}$
$\mathcal{F}_t$ is right cont $\iff \mathcal{F}_t=\mathcal{F}_{t^+}=\bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$ (same fdds)
$\mathcal{F}_t$ is std $\iff \mathcal{F}_0\supset\mathcal{N}_{\mathcal{F}}$ and is right cont
X,Y are equivalent $\iff \forall t_i\in T\ \forall m\in\mathbb{N}\ (X_{t_1},\dots,X_{t_m})\sim(Y\dots)$
vers/modif $\iff \forall t\in T\ X_t=Y_t$ a.s., ie $P(X_t=Y_t)=1$
indistinguish $\iff P(X_t=Y_t\ \forall t\in T)=1 \iff$ ae traj coincide
X is meas $\iff X.(\cdot):(T\times\Omega,\mathcal{B}(T)\otimes\mathcal{F})\to(E,\mathcal{B}(E))$ is a meas map
X prog meas $\iff X.(\cdot):([0,u]\times\Omega,\mathcal{B}([0,u])\otimes\mathcal{F}_u)\to$ // is a
X right/left cont and adapted $\Rightarrow X$ is prog meas $\Rightarrow X$ is meas
* Cdl transformations preserve meas, prog meas and $\perp$

STOPPING TIMES

$\tau:(\Omega,\mathcal{F})\to(T\cup\{+\infty\},\mathcal{B}(\cdot))$ is a stt $\iff \forall t\in T\ (\tau(\omega)\le t)\in\mathcal{F}_t$
$\mathcal{F}_\tau=\{A\in\mathcal{F}_\infty: A\cap(\tau\le t)\in\mathcal{F}_t\ \forall t\}\Rightarrow\sigma(\tau)\subseteq\mathcal{F}_\tau\subseteq\mathcal{F}_\infty\subseteq\mathcal{F}$
σ,τ stt $\Rightarrow \sigma\wedge\tau,\sigma\vee\tau$ stt, $\sigma\le\tau\ \forall\omega\Rightarrow\mathcal{F}_\sigma\subseteq\mathcal{F}_\tau$, $\mathcal{F}_{\sigma\wedge\tau}=\mathcal{F}_\sigma\cap\mathcal{F}_\tau$
$1_{\{\tau(\omega)>(\le)t\}}$ is a prog meas stoch process
X_t prog meas, τ a.s. finite stt $\Rightarrow X_\tau$ is a $\mathcal{F}_\tau$-meas rv
X_t meas, $\tau:(\Omega,\mathcal{F})\to([0,+\infty),\mathcal{B}(\cdot))$ a rv $\Rightarrow X_\tau$ is a $\mathcal{F}$-meas rv



Let $B\in\mathcal{B}(E)$ for X an E-valued proc, then $\left\{\begin{array}{l}B \text{ open, or}\\ B \text{ closed and}\\ \mathcal{F}_t \text{ right cont}\end{array}\right.\Rightarrow \tau_B=\inf(t\ge0: X_t\notin B)$ is a stt
* For the law of a stt start with $F_\tau(t)=P(\tau\le t)$

THE BROWNIAN MOTION

B is a BM $\iff B_0=0$ a.s., $B_t-B_s\perp\mathcal{F}_s$, $B_t-B_s\sim N(0,t-s)$
* 1,2 $\Rightarrow B_t\sim N(0,t)$, 3 $\Rightarrow h(B_t-B_s)\perp\mathcal{F}_s\ \forall h$ cdl function

Properties: (1) $B_t-B_s\perp B_u\ \forall 0\le u\le s$ ie $\perp\sigma(B_u:0\le u\le s)$
(2) B_t is a gaussian proc $\Rightarrow \forall t_i\in T\ \forall m\in\mathbb{N}\ \begin{pmatrix}B_{t_1}\\ \vdots\\ B_{t_m}\end{pmatrix}\sim N(0,\Gamma)$
$\Gamma_{ij}=\mathrm{cov}(B_{t_i},B_{t_j})=E(B_{t_i}B_{t_j})=t_i\wedge t_j$
(3) increments $B_{t_k}-B_{t_{k-1}}$ are $\perp$ and jointly N
(a) $\mathcal{G}_t\subseteq\mathcal{F}'_t\subseteq\mathcal{F}_t\subseteq\ <\mathcal{F}''_t \Rightarrow$ still a BM wrt $\mathcal{F}'_t$ and $\mathcal{F}_t$
(b) $\sigma(B_t-B_s)\perp\mathcal{F}_s$ but not nec wrt $\mathcal{F}''_t$

Constr: B on $(\mathcal{F}_t)$-BM $\Rightarrow$ B is a natural BM $\iff \left\{\begin{array}{l}B_0=0 \text{ a.s.}\\ \forall t_i\in T\ \forall m\in\mathbb{N}\\ E(B_sB_t)=s\wedge t\end{array}\right.$ $\begin{pmatrix}B_{t_1}\\ \vdots\\ B_{t_m}\end{pmatrix}\sim N(0,\Gamma)$

Mobe: $Y_t=\int_0^t X_s\,ds=\lim_{m\to+\infty}\sum_{k=0}^{m-1}X_k\left(\frac{t\cdot k}{m}\right)\left(\frac{t}{m}\right)=\lim X_m$
and $X_m\sim N\Rightarrow Y_t\sim N$ (by a.s. lim $\Rightarrow \mathcal{L}$ lim)

Scaling: $B_{t+s}-B_s$ wrt $\mathcal{F}_{t+s}$, $-B_t$ wrt $\mathcal{F}_t$ (both with stt), $c\cdot B_{t/c^2}$ wrt $\mathcal{F}_{t/c^2}$, $t\cdot B_{1/t}$ wrt $\mathcal{G}_t$ are all BMs (a.s.f)

Variations: $V_\pi^p(f)=\sup_\pi\left[\sum_{i=0}^{m-1}|f(t_{i+1})-f(t_i)|\right]$
* f monotone $\Rightarrow |f(b)-f(a)|$, f $C^1\Rightarrow\int_a^b|f'(t)|\,dt$
$\langle X\rangle_t=(P)\lim_{|\pi|\to0}\left[\sum_{i=0}^{m-1}|f(t_{i+1})-f(t_i)|^2\right]$ F_t of Itô
- $\langle X\rangle_t>0 \iff$ a.s. $V_a^b(X)=+\infty$ | BMs have $\langle B\rangle_t=t$ on
- $\langle X\rangle_t=0 \iff$ a.s. $V_a^b(X)<+\infty$ | $[0,t]$ or $t-s$ on $[s,t]$

Reflection: $P(\tau_a\le t)=P(\sup_{s\in[0,t]}B_t\ge a)=2\cdot P(B_t\ge a)$
$\tau=\inf(t\ge0: X_t\le -a$ or $X_t\ge b)\Rightarrow P(B_\tau=-a)=b/(a+b)$
d-dim: B d-dim BM $\iff$ // a.s., //, $B_t-B_s\sim N(0,(t-s)I)$
and all components $B_k(t)$ are $\perp$ real BMs
$1/t\cdot B_t\xrightarrow{t\to+\infty}0$, $t\cdot B_{1/t}\xrightarrow{t\to0^+}0$

CONDITIONAL EXPECTATION

$\int_C X\,dP=\int_C Z\,dP$
$Z=E(X|\mathcal{D})\iff Z$ is $\mathcal{D}$-meas and (1) $E(X\cdot 1_C)=E(Z\cdot 1_C)$ ∀$C\in\mathcal{D}$ or ∀$C\in\mathcal{E}:\sigma(\mathcal{E})=\mathcal{D}$, or (2) $E(X\cdot Y)=E(Z\cdot Y)$ ∀Y $\mathcal{D}$-meas bdd
Properties: $X\perp\mathcal{D}\iff E[h(X)|\mathcal{D}]=E[h(X)]$ ∀$h\in C_b(\mathbb{R})$ or for just $h(x)=e^{i\langle\lambda,x\rangle}$ ∀$\lambda\in\mathbb{R}^d$, Jen: $h(E)\le E\,h$
Freezing: $E[\Psi(X,\cdot)|\mathcal{D}]=\varphi(x)|_X$, $\varphi(x)=E[\Psi(x,\cdot)]$, X is $\mathcal{D}$-meas, $\Psi(x,\cdot)$ is $\perp\mathcal{D}$. Check $\Psi(X,\cdot)$ and $\varphi(x,\cdot)$ are L^1 wrt
$E[\int_0^t X_s\,ds\,|\,\mathcal{D}]=\int_0^t E(X_s|\mathcal{D})\,ds$ if all is $L^1([0,T])$

MARTINGALE

M_t is a $(\mathcal{F}_t)$-martingale $\iff M_t$ are $\mathcal{F}_t$-meas ∀t, are integrable ∀t so $E(|M_t|)<+\infty$, $E(M_t|\mathcal{F}_s)=M_s$ super $\ge$)($\le$ sub
Doob: M_t right cont, bounded in L^p $\Rightarrow \sup_{t\in T}|M_t|\in L^p$, $E[\sup_t|M_t|^p]\le\left(\frac{p}{p-1}\right)^p\cdot\sup_t[E(|M_t|^p)]$

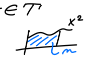
Conv: Let M_t a right cont mart. Then for $p\ge1$
M_t is bdd in L^p $\iff \exists M_\infty\in L^p$, $\mathcal{F}_\infty$-meas $M_t\xrightarrow[t\to+\infty(b^-)]{a.s.,L^p}M_\infty \iff \exists Y\in L^p$, $\mathcal{F}$-meas $M_t=E(Y|\mathcal{F}_t)$
Variation: M_t is a cont $L^2(\mathcal{F}_t)$-mart $\Rightarrow$ (1) ∃! stoch process A_t increasing, cont, adapted, $A_0=0$ a.s. and st $\underbrace{M_t^2-A_t}_{\text{sub-mart}}$ is a mart (2) $A_t=\langle M\rangle_t$
* So non-trivial, cont, L^2 mart dont have finite first variation
Stt: Let M_t a right cont mart
σ,τ a.s. bounded stt $\Rightarrow M_\tau,M_\sigma\in L^1$, $E(M_\tau|\mathcal{F}_\sigma)=M_{\tau\wedge\sigma}$
τ a.s. finite stt $\Rightarrow (M_{t\wedge\tau})_t$ is still a $(\mathcal{F}_t)$-mart
* For $E(\tau)$: use the constant mean property of mart to write $E(M_{t\wedge\tau})=E(M_0)$ and then send $t\to+\infty$ through MCT/DCT
BM: X is a d-dim $(\mathcal{F}_t)$-BM $\iff$ (1) $X_0=0$ a.s. (2) $\forall\lambda\in\mathbb{R}^d$ $Y_t^{(\lambda)}=e^{i\langle\lambda,X_t\rangle+\frac12|\lambda|^2t}$ is a mart
In L^p: $E[|M_t|^p]<+\infty$ ∀t, bdd in L^p: $\sup_{t\in T}E[|M_t|^p]<+\infty$

CHANGE OF PROBABILITY

On $(\Omega,\mathcal{F})$ $Q\ll P\iff P(A)=0\Rightarrow Q(A)=0$ ($\forall A\in\mathcal{F}$), if also $P\ll Q$ then $P\sim Q$
$Q\ll(\sim)P\iff\exists Z\ge(>)0$ $\mathcal{F}$-meas: $dQ=Z\,dP$ and $dP=1/Z\,dQ$
Properties: Let $Q\sim P$, $\mathcal{D}\subseteq\mathcal{F}\Rightarrow dQ|_\mathcal{D}=E(Z|\mathcal{D})\,dP|_\mathcal{D}$, $X\in L^1(Q)\iff X\cdot Z\in L^1(P)$, $X\in L^1(P)\iff X\cdot1/Z\in L^1(Q)$, $\xrightarrow{Q}\iff\xrightarrow{P}$, $E_Q(X|\mathcal{D})=\frac{E_P(X\cdot Z|\mathcal{D})}{E_P(Z|\mathcal{D})}$
$dQ|_{\mathcal{F}_t}=Z_t\,dP|_{\mathcal{F}_t}$
Mart: and $Q|_{\mathcal{F}_t}\sim P|_{\mathcal{F}_t}\Rightarrow Z_t$ or $1/Z_t$ is a $P|_{\mathcal{F}_t}$ or $Q|_{\mathcal{F}_t}$ mart, X_t is a P Q-mart $\iff X_t\cdot Z_t$ ($1/$) is a Q P-mart (and viceversa)

STOCHASTIC INTEGRAL

X_t is $M^p([a,b])$ or $M^p_{loc}([a,b])$ if (1) is prog meas and (2) $E[\int_a^b|X_s|^p\,ds]<+\infty$ or (2loc) $\int_a^b|X_s|^p\,ds<+\infty$
SI, E^2: $\int_a^b X_s\,dB_s=\sum_{k=0}^{m-1}X_k(\omega)(B_{t_{k+1}}-B_{t_k})$
SI, M^2: $\int_a^b X_s\,dB_s=(L^2)\lim_{m\to+\infty}\int_a^b X_m(s)\,dB_s$
$X_m(t)=G_mX(t)=\sum_{k=0}^{m-1}\underbrace{\left[\int_{t_k}^{t_{k+1}}X_s\,ds\right]}_{X_k(\omega)}1_{[t_k,t_{k+1})}(t)$
Properties: Let $\int_a^t X_s\,dB_s$. Then the stoch integral
- wrt $\mathcal{F}_{END\text{-}TIME}=\mathcal{F}_t$ is a meas, $L^2(\Omega,\mathcal{F}_t,P)$ rv
- wrt $\mathcal{F}_{START\text{-}TIME}=\mathcal{F}_a$ we have $E[\int X_s\,dB_s|\mathcal{F}_a]=0$ and $E[(\int X_s\,dB_s)^2|\mathcal{F}_a]=E[\int X_s^2\,ds|\mathcal{F}_a]$ (with the $^2\Rightarrow$ Itô; without the $^2\Rightarrow0$)
- For M^2: $E[\int_a^b h(X_s)\,dB_s\cdot\int_a^b g(Y_s)\,dB_s]=E[\int_a^b h(X_s)\cdot g(Y_s)\,dB_s]=\int_a^b E[h(X_s)g(Y_s)]\,ds$
Conv: $X_m\xrightarrow{M^2}X\iff E[\int_a^b|X_m(s)-X(s)|^2\,ds]\to0$
SI as SP: $\Rightarrow E[\sup_{t\in[0,T]}|\int_0^t X_m(s)-X(s)\,dB_s|^2]\to0$
$X\in M^2([0,T])\Rightarrow I(t)=I_t=\int_0^t X_s\,dB_s$. Is at $I_0=0$ a.s., $I_t-I_s=\int_s^t X_u\,dB_u$, is $\mathcal{F}_t$-meas, is a mart bounded in L^2, is cont/admits a cont version, $\langle I\rangle_t=\int_0^t X_s^2\,ds$
Loc: $X,Y\in M^2$, $X_t(\omega)=Y_t(\omega)$ $\forall\omega\in A\in\mathcal{F}$, $\forall t\in[0,T]$ or $[a,b]$ $\Rightarrow$ their stoch integrals are equal a.s. on A $\forall t\in T$



SI, M^2_{loc}: $I(t)=\int_0^t X_s\,dB_s=(a.s.)\lim_{m\to+\infty}\int_0^t X_m(s)\,dB_s$
$X_m(t)=X_t\cdot1_{[0,\tau_m]}(t)\in M^2$, $\tau_m=\inf(t\in[0,T]:\int_0^t X_s^2\,ds>m)$
Properties: It is a cont $\mathcal{F}_t$-adapted stoch process, $I_0=0$ a.s. In general we lose $\in L^2$, being a mart, Itô isometry
Z on $\mathcal{F}_s$-meas rv $\Rightarrow\int_s^t Z\cdot X_u\,dB_u=Z\cdot\int_s^t X_u\,dB_u$
Example: $\int_s^t 1\,dW_u=W_t-W_s$, $\int_0^t W_s\,dW_s=\frac12W_t^2-\frac t2$
* Stoch integrals of deterministic functions are gaussian, $\sim N(0,\sigma^2)$ with σ^2 computable through Itô isometry.

LOCAL MARTINGALE

M_t is a local mart $\iff\exists\tau_n$ a sequence of stt st (1) $\tau_n\nearrow+\infty$ a.s. and (2) $M_{t\wedge\tau_n}$ is an $(\mathcal{F}_t)$-mart $\forall n$
Variation: M_t a cont local mart $\Rightarrow$ (1) ∃! stoch process A_t st … and $M_t^2-A_t$ is a cont local mart (2) $A_t=\langle M\rangle_t$
SI: $X\in M^2_{loc}\Rightarrow I_t=\int_0^t X_s\,dB_s$ is a local mart (and $\tau_n=\inf(\dots>n)$ is the canonical one) and $\langle I\rangle_t=\int_0^t X_s^2\,ds=A_t$
Co. Var: M,N are cont local mart $\Rightarrow$ ∃! stoch process A_t with $V_0^tA<+\infty$ and st $M_tN_t-A_t$ is a cont local mart where $A_t=\langle M,N\rangle_t=1/4(\langle M+N\rangle_t-\langle M-N\rangle_t)$

STOCHASTIC CALCULUS

X is an Itô process $\iff\exists F\in M^1_{loc}([0,T])$ (drift), $\exists G\in M^2_{loc}([0,T])$ (diffusion) st
$X_t=X_0+\int_0^t F_s\,ds+\int_0^t G_s\,dB_s\rightsquigarrow dX_t=F_t\,dt+G_t\,dB_t$. That also
$\Rightarrow X$ is a.s. a cont process, $X\in M^p_{loc}([0,T])$ $\forall p\ge1$, F and G ∃!
Variation: $\langle X\rangle_t=\int_0^t G_s^2\,ds$, $\langle X,Y\rangle_t=\int_0^t G_1(s)\cdot G_2(s)\,ds$ $\begin{array}{c|cc} & & dt\end{array}$
Leibniz: X,Y Itô $\Rightarrow d(X_tY_t)=X_t\,dY_t+Y_t\,dX_t+d\langle X,Y\rangle_t$ $dX_t\cdot dY_t$
Itô formula: X Itô, $f\in C^{2,1}(\mathbb{R}\times\mathbb{R}^+;\mathbb{R})\Rightarrow Y_t=f(X_t,t)$ is Itô and $dY_t=(\partial_tf)\,dt+(\partial_xf)\,dX_t+\frac12(\partial_{xx}f)\,d\langle X\rangle_t$, $f|_{(X_t,t)}$
Mart: $F_t=0$ $\forall t\Rightarrow X_t$ local mart, if more $G_t\in M^2\Rightarrow X_t$ mart $\Big\}$ if we have also that $X_0\in L^1(\Omega)$
* For checking $G_t\in M^2$ fix $T\in[0,+\infty)$ and estimate $E[\int_0^T\dots]$
* For Itô processes $V_0^tX=\int_0^t|F_s|\,ds$ (if $\langle X\rangle_t=0$)
* Deterministic stuff (and normal AN1 functions) have $0\,dB_t$
M_t local mart, $>0\Rightarrow$ super mart $\Rightarrow$ mart if const E
Multi dim: X_t m-dim is Itô if really as before but $F_i(m\times1)$, $G_{ij}(m\times d)\Rightarrow X_i(t)=X_i(0)+\int_0^tF_i\,ds+\sum_{j=1}^d\int_0^tG_{ij}\,dB_j(s)$
$\langle X_i\rangle_t=\sum_{k=1}^d\int_0^tG_{ik}^2(s)\,ds$, $\langle X_i,X_j\rangle_t=\sum_k\int_0^tG_{ik}G_{jk}\,ds$
Itô formula: $f\in C^{2,1}(\mathbb{R}^m\times\mathbb{R}^+;\mathbb{R})\Rightarrow Y_t=f(X_t,t)$ is Itô,
$dY_t=(\partial_tf)\,dt+\sum(\partial_{x_i}f)\,dX_i(t)+1/2\sum\sum(\partial^2_{x_ix_j}f)\,d\langle X_i,X_j\rangle_t$
Thm: $X_1,X_2\in M^2([0,T])$ and B_1,B_2 components of a d-dim BM $\Rightarrow E[\int_0^tX_1\,dB_1\cdot\int_0^tX_2\,dB_2\,|\,\mathcal{F}_s]=0$, $I_i(t)=\int_0^tX_i\,dB_i$ are st $I_1(t)\cdot I_2(t)$ is a mart, and $\langle I_1,I_2\rangle_t=0$.

GIRSANOVS THM & MART REP

Let B a std and cont m-dim BM. Then $Z_t=\exp(\int_0^t\varphi_s\,dB_s-1/2\int_0^t\varphi_s^2\,ds)$ is a mart, and cont, bdd in L^p $\forall p\ge1$ and st $E(Z_t)=1$ $\forall t\in[0,T]$ $\iff\varphi\in M^2([0,T])$
local mart and $>0\Rightarrow$ super $\iff$ $\varphi\in M^2_{loc}([0,T])$ $E(Z_T)=E(Z_0)=1$
thm $\Rightarrow$ $Q: dQ=Z_T\,dP|_{\mathcal{F}_T}\iff Q\sim P|_{\mathcal{F}_T}$
$\Rightarrow W_t=B_t-\int_0^t\varphi_s\,ds$ is a cont and std $(\mathcal{F}_t)$-BM wrt Q $\quad|\varphi_s|^2=\|\varphi_s\|^2$
* To find φ_s: use $\exp(\sigma B_t-1/2\sigma^2t)$ is a mart if B_t is a BM. Rewrite dX_t w $dB_t\rightsquigarrow dW_t$, to have X a local m.
* To check Z_t is a mart: $E[e^{1/2\int_0^T|\varphi_s|^2ds}]<+\infty$, $\exists\eta>0$: $\forall s\in[0,T]$ $E[e^{\eta|\varphi_s|^2}]<+\infty$, $\varphi\in M^2([0,T])$ and $E[e^{\int_0^T1/2\,\varphi_s\,dB_s}]<+\infty$
$\Rightarrow$ (ui, 1, ui) mart, ui $\iff$ bdd L^1, Novikov, _, Kazamaki crit
Mart rep: M_t a $(\overline{\mathcal{G}}_t)^{BM}$-$L^2$-mart $\Rightarrow M_t=m+\int_0^tH_s\,dB_s$
* $m\in\mathbb{R}\rightsquigarrow$ through E, $H_t\in M^2([0,T])\rightsquigarrow$ through Itô for dM_t
* That rep holds act $\forall X_t\in L^2$, let $Y_t=E(X|\overline{\mathcal{G}}_t)$, _, eval at $t=T$
(Exp) Mart rep: Find φ_t: $M_t=Z_t$ above. Let U_t: $U_tM_t=1$) that is $\iff d(U_tM_t)=0$, and from that get conds for φ_t. $|_{t=0}$

SDEs $\left\{\begin{array}{l}dX_t=b(t,X_t)\,dt+\sigma(t,X_t)\,dB_t\\ X(0)=\eta\sim\mathcal{L}\text{ or }x_0\in\mathbb{R}\end{array}\right.$ $\Big|$ $\begin{array}{l}b:[0,T]\times\mathbb{R}^m\to\mathbb{R}^m\\ \sigma:[0,T]\times\mathbb{R}^m\to M(m\times d)\end{array}$

Ass A (A'): (1) are jointly meas wrt (t,x), (2) sublinear growth, $\exists M>0:|f(t,x)|\le M(1+|x|)$ $\forall x\in\mathbb{R}^m$, (3) locally Lipschitz, $\forall N>0$ $\exists L_N>0:|f(t,x)-f(t,y)|\le L_N\cdot|x-y|$ $\forall x,y\in\mathbb{R}^m$: $\|x\|\le N$, $\|y\|\le N$

M3: Collect X_t. Move it to LHS, identify the diff, compute it w/ Itô.
M2: Robust method. When we have $\sigma(t)X_t\,dB_t$. Introduce $U_t=\exp(-\int_0^t\sigma(s)\,dB_s+1/2\int_0^t\sigma^2(s)\,ds)$ and $Y_t=U_tX_t$. Show are Itô, compute d-t, from dY_t see she solves an ODE w/ locally Lip and sublinear coeffs, recover $X_t=Y_tU_t^{-1}$
M1: Var of const. When $dX_t=\alpha(t)\,dt+\beta(t)X_t\,dt+\sigma(t)\,dB_t$. Solve the homogen, $Y_t=x_0\cdot e^{\int_0^t\beta(s)\,ds}$, set $X_t=\tilde X_te^{\lambda(t)}$, characterize/find $\tilde X_t$ w/ $d\tilde X_t$, recover X_t. The sol is $X_t=e^{\lambda(t)}\tilde X_t=e^{\lambda(t)}[x_0+\int_0^te^{-\lambda(s)}\alpha(s)\,ds+\int_0^te^{-\lambda(s)}\sigma(s)\,dB_s]$

Geom BM: $dX_t=bX_t\,dt+\sigma X_t\,dB_t\Rightarrow X_t=x_0\cdot e^{[(b-\frac12\sigma^2)t+\sigma B_t]}$
BBridge: $dX_t=\frac{-1}{1-t}X_t\,dt+dB_t\Rightarrow X_t=(1-t)\int_0^t\frac{1}{1-s}\,dB_s$
* To prove two process are indistinguishable (we to have path wise!) check $d(\tilde Z_t\cdot1/Z_t)=0$ $\forall t\Rightarrow\tilde Z_t\cdot1/Z_t=$ const ? 1
Markov: Ass A' $\Rightarrow X_t$ Markov, b and $\sigma\perp t\Rightarrow$ homogeneous
For L: compute $df(X_t)$, $f\in C_b^2$, and $L_uf(x)$ is $[\dots]\,dt$
FK: $\left\{\begin{array}{l}\partial_t\Psi=L_t\Psi-c\Psi+f\\ \Psi(0,x)=\varphi(x)\end{array}\right.$ (FW) $\left\{\begin{array}{l}-\partial_t\Psi=L_t\Psi-c\Psi+f\\ \Psi(T,x)=\varphi(x)\end{array}\right.$ (BW)
$\Psi(t,x)=E^{0,x}\left[\varphi(X_t)e^{-\int_0^tc(u,X_u)\,du}+\int_0^tf(s,X_s)e^{-\int_0^tc(u,X_u)\,du}\,ds\right]$ $\quad t,x$ $\quad\varphi(X_T)e^{-\int_t^T}$ $\quad\int_t^T$ $\quad e^{-\int_t^T}$